

# Condensation of an ideal gas obeying non-Abelian statistic

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We consider the thermodynamic geometry of an ideal non-Abelian gas. We show that, for a certain value of the fractional parameter and at the relevant maximum value of fugacity, the thermodynamic curvature has a singular point. This indicates a condensation such as Bose-Einstein condensation for non-Abelian statistics and we work out the phase transition temperature in various dimensions.

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## I. INTRODUCTION

Bose-Einstein condensation (BEC) is a well-known phenomenon for boson gas which was first predicted in 1924 [1] and later observed in 1995 in a remarkable series of experiments on vapors of rubidium and sodium [2, 3]. It has been shown that there is no condensation for the nonrelativistic boson particles in the two and one spatial dimensions while, for ultra relativistic particles one can find a finite phase transition temperature in all dimensions  $D \geq 2$  [4]. A natural question to raise here is whether there are other statistical systems that can be condensate and if they occur in low dimensions. It has been shown that an ideal boson gas has an upper bound on fugacity, where the thermodynamic curvature is singular and where BEC occurs. Recently, we investigated the thermodynamic geometry of some systems with intermediate statistics and found some evidence for the condensation of nonpure bosonic systems.

Thermodynamics of different models can be studied by a qualitative tool, namely, the thermodynamic curvature, which has been introduced by the theory of thermodynamic geometry [5–7]. Thermodynamic geometry can be used as a measure of statistical interaction, e.g., the thermodynamic curvature of an ideal classical gas is zero and it is positive and negative for boson and fermion gases, respectively, indicating that the statistical interactions of these models are non interacting, attractive, and repulsive [8, 9]. It was argued that the scalar curvature could be used to show that fermion gases were more stable than boson gases and that, therefore, one can utilize the scalar curvature as a stability criterion [9]. Also, it has been shown that the singular point of thermodynamic curvature coincides with the critical phase transition point of some thermodynamic systems [9, 10]. Recently, we worked out the thermodynamic curvature of a thermodynamic system in which particles obey fractional exclusion statistics [11]. The study yielded some interesting results about the thermodynamic properties of these systems.

Investigation of the thermodynamic geometry of an ideal gas with particles obeying Abelian fractional exclusion statistics has shown that there is no phase transition such as Bose-Einstein condensation, while for other kinds of intermediate

statistics such as Polychronakos and  $q$ -deformed bosons, the singular point of thermodynamic curvature coincides with the condensation point [11]. In this paper, we consider the possibility of condensation of an ideal gas with particles obeying non-Abelian exclusion statistics. A hint to a possible origin of non-Abelian systems may lie in the observation that these systems are fluids in which microscopic particles of a quantum statistics group form clusters whose statistics is effectively bosonic. These clusters then form a Bose-Einstein condensate. Generally, breaking of a cluster in that condensate costs energy equal at least to the energy gap. Generally speaking, vortices formed in these Bose-Einstein condensates become non-Abelian quasiparticles when their presence allows clusters to be broken at no energy cost, leading to ground state degeneracy [12].

Haldane introduced the idea of generalized Pauli principle or fractional exclusion statistics [13]. His definition of particles obeying the generalized exclusion statistics with finite dimensional Hilbert space is motivated by such physical examples as quasiparticles in the fractional quantum Hall system and spinons in antiferromagnetic spin chains [14]. The fractional parameter governing the fractional exclusion statistics in this case is defined by  $g = -\frac{\Delta d}{\Delta N}$ , where  $\Delta d$  is the change in the dimension of the single particle Hilbert space and  $\Delta N$  is the change in the number of particles when the size of the system and the boundary conditions are unchanged. The fractional parameter is a real number  $0 \leq g \leq 1$ , and the marginal value  $g = 0$  ( $g = 1$ ) corresponds to bosons (fermions). It has been shown that spinons in the Heisenberg chains could also show the non-Abelian exclusion statistics [15]. It is well known that strongly correlated many body systems, especially in the low dimensions, can have quasiparticles that are very different from the microscopic degrees of freedom from which the system is built. Also, the braid group in two spatial dimensions may be represented by the noncommute matrices; the non-Abelian braid statistics could be realized in a real world. These include the quasi particles in the non-Abelian fractional quantum Hall state as well as those in the conformal field theory [16, 18].

This paper is organized as follows. In Sec. II, we summarized some properties of the non-Abelian statistics and some thermodynamic quantities of an ideal gas with particles obeying non-Abelian statistics being explored. Thermodynamic geometry and some interesting properties of this kind of statistics are investigated in Sec. III. In Sec. IV, we will show that the singular point of thermodynamic geometry at the up-

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per bound of fugacity coincides with to the singular point of derivative of particle number and there is a condensation at a related temperature for a specified value of fractional parameter.

## II. NON-ABELIAN STATISTICS

Using the thermodynamic Bethe ansatz for specific nondiagonal scattering matrices, the occupation number distribution function for the particle obeying the non-Abelian statistics has been derived by Guruswamy and Schoutens [17]. They proposed the following equation for  $M$  types of particles and  $k$  types of pseudoparticles:

$$\left(\frac{\Lambda_A - 1}{\Lambda_A}\right) \prod_B \Lambda_B^{\alpha_{AB}} \prod_i \Lambda_i^{G_{Ai}} = z_A, \quad A = 1, \dots, M, \\ \left(\frac{\Lambda_i - 1}{\Lambda_i}\right) \prod_A \Lambda_A^{G_{iA}} \prod_j \Lambda_j^{G_{ij}} = 1, \quad i = 1, \dots, k-1, \quad (1)$$

where  $\alpha_{AB}$  is the Abelian part statistics matrix,  $G_{iA} = G_{Ai} = -\frac{1}{2}\delta_{i,1}$  and  $G_{ij} = \frac{1}{2}(C_{k-1})_{ij}$ , with  $C_{ij}$  as the Cartan matrix of the associated group. Also,  $\Lambda_a$  is the single-level grand canonical partition function and  $z_a = \exp[\beta(\mu_a - \varepsilon)]$  with  $\varepsilon$  as the energy level and  $\mu_a$  representing the chemical potential of the particle of type  $a$ . In the case  $G_{iA} = 0$ , the above equations describe the Haldane fractional exclusion statistics. The simplest extension of the Abelian to non-Abelian case is related to the  $M = 1$  and  $k = 2$ , which can describe the  $q$ -pfaffian non-Abelian fractional quantum Hall effect. In this case, the above equations reduce to the following relations [16, 17]:

$$\left(\frac{\Lambda - 1}{\Lambda}\right) \Lambda^\alpha \Lambda_1^{-1/2} = z, \quad \left(\frac{\Lambda_1 - 1}{\Lambda_1}\right) \Lambda^{-1/2} \Lambda_1 = 1. \quad (2)$$

Eliminating  $\Lambda_1$  from the above equations yields,

$$\left(\frac{\Lambda - 1}{\Lambda}\right) \Lambda^\alpha (1 + \Lambda^{1/2})^{-1/2} = z. \quad (3)$$

Since the distribution function  $n$  is defined by  $n = \frac{d \ln(\Lambda)}{d \ln(z)}$ , we obtain

$$n(\Lambda) = \frac{1}{\frac{1}{4} \frac{1}{1 + \Lambda^{1/2}} + \frac{1}{\Lambda - 1} + (\alpha - \frac{1}{4})}. \quad (4)$$

It is simple to show that the distribution function  $n(\Lambda)$  has two different behaviors in the extremal limits of  $z$  which are not exactly similar to the Abelian exclusion statistics.

$$n = \begin{cases} \frac{1}{\alpha - \frac{1}{4}} & \alpha > \frac{1}{4}, z \rightarrow \infty \\ \infty & \alpha \leq \frac{1}{4}, z \rightarrow z_{\max} \end{cases}, \quad (5)$$

For  $\alpha > \frac{1}{4}$ , the maximum population  $\frac{1}{\alpha - \frac{1}{4}}$  is finite and we do not expect a phase transition. This is similar to the Abelian fractional exclusion statistics where  $n(\Lambda) \sim \frac{1}{g}$  ( $z \rightarrow \infty$ ,  $g =$

$\alpha - \frac{1}{4}$ ). However, for  $\alpha \leq \frac{1}{4}$  behavior of  $n(\Lambda)$  is similar to a bosonic gas and so we cannot exclude condensed states. This is an interesting possibility for this non-Abelian statistics. It should be noted that in this case the non-Abelian statistics has a completely different behavior compared to the Abelian fractional exclusion statistics.

In the thermodynamic limit, the internal energy and particle number of an exclusion gas in a  $D$ -dimensional box of volume  $L^D$  with the dispersion relation  $\varepsilon = ap^\sigma$  can be written as

$$U = \int_0^\infty \varepsilon n(\Lambda) \Omega(\varepsilon) d\varepsilon, \quad N = \int_0^\infty n(\Lambda) \Omega(\varepsilon) d\varepsilon. \quad (6)$$

Neglecting the spin of the particles,  $\Omega(\varepsilon) = \frac{A^D}{\Gamma(\frac{D}{\sigma})} \varepsilon^{D/\sigma - 1}$  will be the density of the single particle state for the system.  $A = \frac{L\sqrt{\pi}}{a^{1/\sigma} h}$  is a constant and we will set it equal to one ( $A = 1$ ) for simplicity. Obtaining the internal energy and the particle number of the ideal non-Abelian gas for an arbitrary fractional parameter will not be possible because Eq. (3) is not solvable with respect to  $\Lambda$  for an arbitrary value of fractional parameter. One can change the integrating variable  $\varepsilon$  to  $\Lambda$  by using the following relations

$$\varepsilon = \frac{1}{\beta} \ln \left[ \frac{z\sqrt{1 + \sqrt{\Lambda}}}{(\Lambda - 1)\Lambda^{\alpha-1}} \right], \quad \frac{d}{d\varepsilon} \ln \Lambda = -\beta n(\Lambda), \quad (7)$$

where  $z = \exp(\beta\mu)$  is the fugacity of gas. Thus, the internal energy and particle number are given by

$$U = -\frac{\beta^{-(D/\sigma)-1}}{\Gamma(\frac{D}{\sigma})} \int_{\Lambda_0}^1 \left\{ \ln \left[ \frac{z\sqrt{1 + \sqrt{\Lambda}}}{(\Lambda - 1)\Lambda^{\alpha-1}} \right] \right\}^{D/\sigma} d(\ln \Lambda), \\ N = -\frac{\beta^{-D/\sigma}}{\Gamma(\frac{D}{\sigma})} \int_{\Lambda_0}^1 \left\{ \ln \left[ \frac{z\sqrt{1 + \sqrt{\Lambda}}}{(\Lambda - 1)\Lambda^{\alpha-1}} \right] \right\}^{(D/\sigma)-1} d(\ln \Lambda), \quad (8)$$

where  $\Lambda_0$  is the zero energy grand partition function which satisfies  $z\sqrt{1 + \sqrt{\Lambda_0}} = (\Lambda_0 - 1)\Lambda_0^{\alpha-1}$  and  $\varepsilon = \infty$  corresponds to  $\Lambda = 1$ .

## III. THERMODYNAMIC GEOMETRY OF NON-ABELIAN STATISTICS

Now, we are going to construct the thermodynamic geometry of an ideal non-Abelian statistic gas. One can use various thermodynamic potentials for obtaining the metric elements of the thermodynamic parameter space. The appropriate representation in the present case belongs to the second derivatives of the logarithm of partition function,  $\mathcal{Z}$ , with respect to the thermodynamic parameters  $\beta = 1/k_B T$  and  $\gamma = -\mu/k_B T$ . Therefore, the metric elements are given by

$$g_{ij} = \frac{\partial^2 \ln \mathcal{Z}}{\partial \beta^i \partial \beta^j}. \quad (9)$$

The thermodynamic parameter space is a two-dimensional (2D) space  $(\beta^1, \beta^2) = (\beta, \gamma)$ . For computing the thermodynamic metric, we select one of the extended variables as

the constant system scale. We will implicitly pick volume in working with the grand canonical ensemble. The metric elements of thermodynamic space of an ideal non-Abelian statistic gas are given by

$$\begin{aligned}
G_{\beta\beta} &= \frac{\partial^2 \ln \mathcal{Z}}{\partial \beta^2} = -\left(\frac{\partial U}{\partial \beta}\right)_\gamma \\
&= \frac{(\frac{D}{\sigma} + 1)\beta^{-(D/\sigma+2)}}{\Gamma(\frac{D}{2})} \int_{\Lambda_0}^1 \left\{ \ln \left[ \frac{z\sqrt{1+\sqrt{\Lambda}}}{(\Lambda-1)\Lambda^{\alpha-1}} \right] \right\}^{D/\sigma} d(\ln \Lambda), \\
G_{\beta\gamma} &= \frac{\partial^2 \ln \mathcal{Z}}{\partial \gamma \partial \beta} = -\left(\frac{\partial N}{\partial \beta}\right)_\gamma \\
&= \frac{\frac{D}{\sigma}\beta^{-(D/\sigma+1)}}{\Gamma(\frac{D}{2})} \int_{\Lambda_0}^1 \left\{ \ln \left[ \frac{z\sqrt{1+\sqrt{\Lambda}}}{(\Lambda-1)\Lambda^{\alpha-1}} \right] \right\}^{\frac{D}{\sigma}-1} d(\ln \Lambda), \quad (10) \\
G_{\gamma\gamma} &= \frac{\partial^2 \ln \mathcal{Z}}{\partial \gamma^2} = -\left(\frac{\partial N}{\partial \gamma}\right)_\beta \\
&= \frac{\beta^{-D/\sigma}}{\Gamma(\frac{D}{2})} \int_{\Lambda_0}^1 \frac{1}{n} \frac{\partial n}{\partial \Lambda} \frac{\partial \Lambda}{\partial \gamma} \left\{ \ln \left[ \frac{z\sqrt{1+\sqrt{\Lambda}}}{(\Lambda-1)\Lambda^{\alpha-1}} \right] \right\}^{(D/\sigma)-1} d(\ln \Lambda).
\end{aligned}$$

Now we can evaluate the Christoffel symbol,  $\Gamma_{ijk} = \frac{1}{2}(g_{ij,k} + g_{ik,j} - g_{jk,i})$ , using the derivative of the metric elements with respect to the thermodynamic parameter. Therefore, one can evaluate the well-known Riemann tensor, Ricci tensor, and finally, the Ricci scalar, which will be called the thermodynamic curvature because of the identity of the constructed geometry.

Before considering the thermodynamic geometry of the system, we explain a subtle point about the fugacity of an ideal non-Abelian statistic gas. We remember that the fugacity of an ideal boson gas has an upper bound, where the Bose-Einstein condensation occurs. In a recent paper, we showed that the fugacity of an ideal gas with particles obeying a special kind of fractional statistics has an upper bound, too, which is related to the fractional parameter values [11]. We have investigated the thermodynamic geometry of Abelian Haldane fractional exclusion gas in more detail to find no upper bound on fugacity and, therefore, no condensation to occur. Here, we consider the existence of the upper bound on fugacity of a non-Abelian gas. For zero energy particles, the right hand side of Eq. (3) is replaced by the fugacity of gas. Exploring a condition for the maximum value of fugacity motivated us to differentiate Eq. (3) with respect to  $\Lambda$ . Thus,  $\frac{dz}{d\Lambda} = 0$  represents a relation between the fractional parameter and  $\Lambda$ ,

$$\Lambda = \frac{32\alpha^2 - 40\alpha + 9 + \sqrt{64\alpha^2 - 80\alpha + 17}}{2(4\alpha - 1)^2}, \quad (11)$$

Substituting the above equation in Eq. (3) reveals that the maximum point appears only in the interval  $0 \leq \alpha \leq 0.25$  as depicted in Fig. 1. For other values of fractional parameter out of this interval, there is no upper bound on fugacity. The thermodynamic geometry of an ideal non-Abelian gas is presented in Fig. 2. One can observe that the thermodynamic curvature of a non-Abelian gas has a singularity for fractional parameters in the interval  $0 \leq \alpha \leq 0.25$  at relevant maximum values of fugacity. It should be mentioned that the statistical

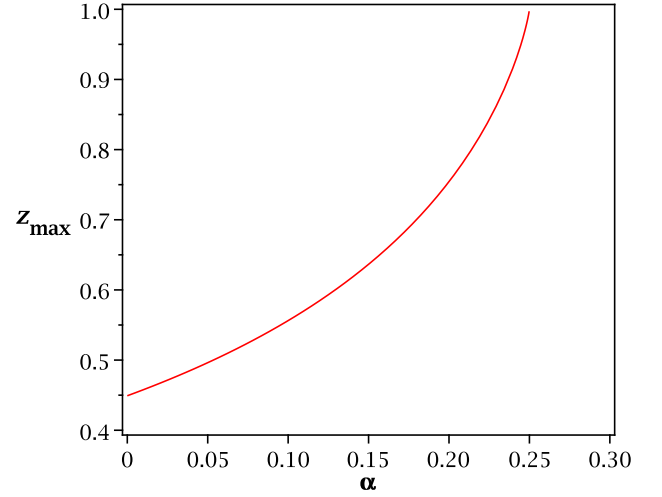


FIG. 1: (Color online) Maximum value of fugacity of an ideal non-Abelian gas as a function of fractional parameter.

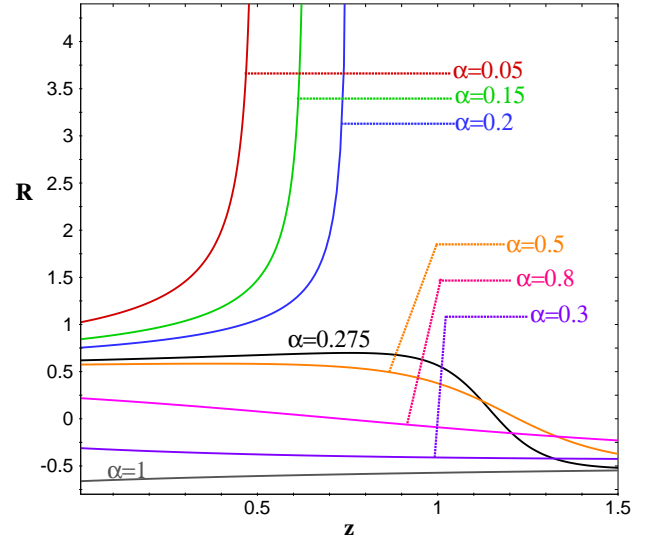


FIG. 2: (Color online) Thermodynamic curvature of an ideal non-Abelian gas as a function of fugacity for isotherm processes ( $\beta = 1, D/\sigma = 3/2$ ).  $R$  is singular at  $z = z_{\max}$  for fractional parameter in the interval  $0 \leq \alpha \leq 0.25$ .

interaction of an ideal non-Abelian gas is attractive in the full physical range for the fractional parameter in the specified interval, while for other values of fractional parameter, the thermodynamic curvature tends toward negative values at the low temperature limit (quantum limit) and the statistical interaction becomes repulsive. It seems that the particles obeying non-Abelian fractional statistics with the fractional parameter  $\alpha > 0.25$  at  $T = 0$  exhibit Fermi surfaces such as Haldane fractional exclusion statistics [11].

#### IV. CONDENSATION OF NON-ABELIAN STATISTIC GAS

It is well known that the thermodynamic curvature of an ideal boson gas is singular at  $z = 1$ . Also, the particle number as a function of fugacity has an infinite slope at the maximum value of fugacity where the Bose-Einstein condensation occurs. In the case of ideal non-Abelian gas, it is evident from Figs. 3 and 4 that the particle number for each value of the fractional parameter in the foregoing interval has an infinite slope and a finite value at the relevant maximum point of fugacity. In other words, we can use an analytical method to show that the derivative of the particle number with respect to fugacity is singular at the maximum value of fugacity, which indicates that the maximum value is a critical value of fugacity. According to Eq. (6) the derivative of particle number is derived

$$\frac{\partial N}{\partial z} = \int_0^\infty \frac{\partial n(\Lambda)}{\partial \Lambda} \frac{\partial \Lambda}{\partial z} \Omega(\epsilon) d\epsilon. \quad (12)$$

Since  $\frac{\partial z}{\partial \Lambda}|_{z_{\max}} = 0$ , derivative of the particle number diverges at maximum value of fugacity. Therefore, we argue that there is a condensation for a non-Abelian gas at maximum value of fugacity for the fractional parameter in a specific interval. It has been shown that the phase transition temperature for an ideal boson gas has a finite value for  $\frac{D}{\sigma} > 1$ . Now we explore the possibility of condensation in various cases for an ideal non-Abelian gas. One can obtain the phase transition temperature by using the particle number relation at  $z = z_{\max}$  in Eq.

(8),

$$k_B T_c = \frac{ah^\sigma}{\pi^{\sigma/2}} \left( - \frac{\int_{\Lambda_0}^1 \left\{ \ln \left[ \frac{z_{\max} \sqrt{1+\sqrt{\Lambda}}}{(\Lambda-1)\Lambda^{\alpha-1}} \right] \right\}^{(D/\sigma)-1} d(\ln \Lambda)}{n\Gamma(D/2)} \right)^{-\sigma/D} \quad (13)$$

where  $T_c$  is the phase transition temperature and  $n = \frac{N}{L^D}$  is the particle density assumed to be constant. Figure 5 shows that there is a finite phase transition temperature in three and two spatial dimensions for a nonrelativistic, ideal non-Abelian gas with a fractional parameter in the interval mentioned above. In the two dimensional case, for  $\alpha = 0.25$ , the phase transition temperature tends to zero,  $T_c = 0$ , while the maximum value of fugacity is  $z_{\max} = 1$ . This point is similar to a boson gas which has no condensation in two dimensions for nonrelativistic particles. For ultrarelativistic particles, one can find a finite critical temperature in all dimensions except in  $D = 1$  for  $\alpha \geq 0.25$ . It is an interesting phenomenon that, unlike the case of the boson gas, the condensation for a non-Abelian gas can occur in even two spatial dimensions with the particles in a nonrelativistic regime but in all dimensions in an ultrarelativistic regime.

The interval of fractional parameter in which there is an upper bound on fugacity may be related to the selection of  $M$  types of particles and  $k$  types of pseudoparticles. We selected the simplest form of non-Abelian statistics. For the other values of  $M$  and  $k$ , we encountered a certain degree of complexity when trying to obtain the distribution function and the ensuing evaluations. To answer the initial question of this paper, selection of the simplest case led us to the non-Abelian exclusion statistics which could be condensate in even low dimensions.

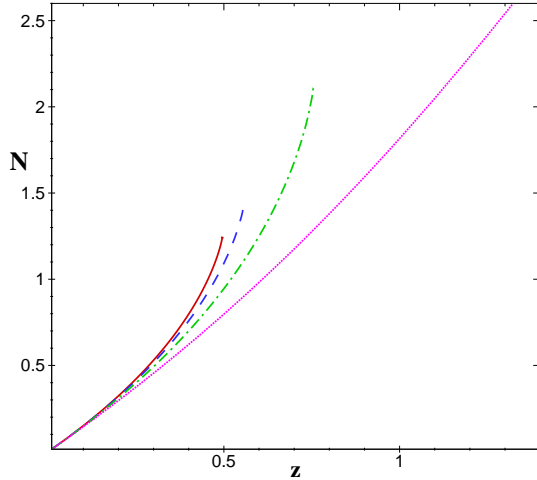


FIG. 3: (Color online) Particle number of an ideal non-Abelian gas with different fractional parameter (red solid line,  $\alpha = 0.05$ ; blue dashed line  $\alpha = 0.1$ ; green dash-dotted line  $\alpha = 0.2$ ; and purple dotted line,  $\alpha = 0.4$ ) as a function of fugacity for isotherm processes. ( $\beta = 1$ ,  $D/\sigma = 3/2$ ).

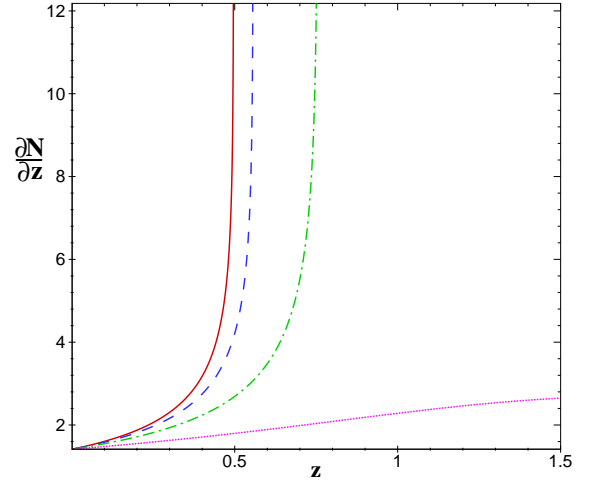


FIG. 4: (Color online) The derivative of particle number of an ideal non-Abelian gas with different fractional parameter (red solid line,  $\alpha = 0.05$ ; blue dashed line  $\alpha = 0.1$ ; green dash-dotted line  $\alpha = 0.2$ ; and purple dotted line  $\alpha = 0.4$ ) as a function of fugacity for isotherm processes. ( $\beta = 1$ ,  $D/\sigma = 3/2$ ).

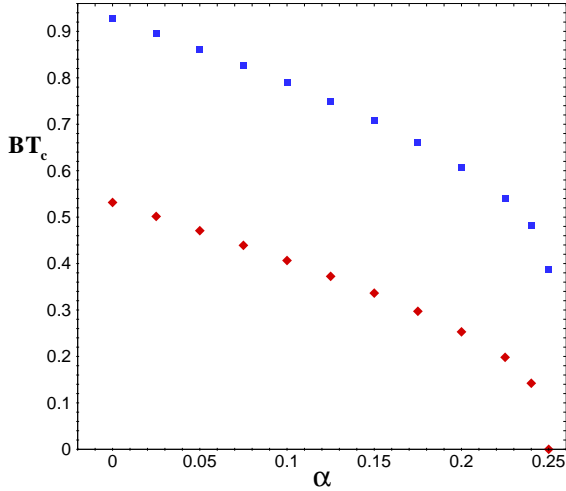


FIG. 5: (Color online) Phase transition temperature as a function of fractional parameter for non-relativistic particles in ( $D = 3$ ) (blue square symbols) and ( $D = 2$ ) (red diamond symbols).  $B = \frac{k_B \pi}{ah^2 n^{2/D}}$  is a constant.

## V. CONCLUSION AND DISCUSSION

We considered the non-Abelian statistics, which has  $k = 2$ . We notice that if one sets  $G_{IA} = 0$ , which is the non-Abelian part of the statistics, we will have

$$(\Lambda - 1)\Lambda^\alpha = z. \quad (14)$$

The above equation describes the non-Abelian statistics, where  $\alpha$  is a parameter and takes values  $0 \leq \alpha \leq 1$ . As we know, the Moore-Read state in addition to  $k = 2$  has a free parameter  $M = q - 1$ , which set the filling fraction,  $\nu = k/(Mk + 2) = 1/(M + 1)$ , and the related fractional pa-

rameter is  $\alpha = [(k - 1)M + 2]/2(kM + 2) = (M + 2)/4(M + 1)$ , which is bounded from below by  $1/4$  [17]. Therefore, Eq. (3) for  $\alpha > 1/4$  describes the Moore-Read states and according to the behavior of thermodynamic curvature and mean occupation number in the low temperature limit, the statistics is fractional exclusion such as Haldane statistics, and therefore there is no condensation for  $\alpha > 1/4$ . We have also explored  $0 \leq \alpha \leq 1/4$  where the theory is still well defined and consistent, however, the non-Abelian statistics is boson like and has no exclusion property. The statistical interaction is attractive in full physical range and it has singularity for specified values of fugacity where the condensation occurs. It should be noted that the non-Abelian character of the gas for  $0 \leq \alpha \leq 1/4$  should be investigated more carefully in the future and we cannot rely on the results from fractional exclusion statistics.

Thermodynamic geometry of an ideal non-Abelian statistic gas worked out and one can observe that for  $\alpha \leq 1/4$ , there is an upper bound for fugacity of gas, where the thermodynamic curvature is singular. Also, the particle number has a finite value and infinite slope at  $z = z_{\max}$ . Therefore, we argued that there is a phase transition such as Bose-Einstein condensation for specified values of fractional parameter. Phase transition temperature was derived and it was shown that there is a finite temperature, even for low dimensions.

Whereas the non-Abelian statistics has many interesting relationships with the fractional quantum Hall effect and the non-Abelians play a major role in the  $q$ -pfaffian Hall states [18] and also due to their candidacy for constructing the topological quantum computers [19], the condensation of these kinds of non-Abelian statistics in 3D and 2D can be an outstanding phenomenon in this field and may point to new states of matter.

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- [1] S. N. Bose, Z. Phys. 26, 178, (1924); A. Einstein, Sitzber. Kgl. Preuss. Akad. Wiss. 261 (1924).
  - [2] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman and E. A. Cornell, Science **269**, 198 (1995).
  - [3] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).
  - [4] R. M. May, Phys. Rev. **135**, A1515 (1964).
  - [5] L. Tisza, *Generalized Thermodynamics*, (M.I.T, Cambridge, MA, 1966).
  - [6] F. Weinhold, J. Chem. Phys. **63**, 2479 (1975).
  - [7] G. Ruppeiner, Phys. Rev. A **20**, 1608 (1979).
  - [8] J. D. Nulton and P. Salamon, Phys. Rev. A **31**, 2520 (1985).
  - [9] H. Janyszek and R. Mrugała, J. Phys. A: Math. Gen. **23**, 467 (1990); **23**, 477 (1990); Phys. Rev. A **39**, 6515 (1989).
  - [10] D. C. Brody and D. W. Hook, J. Phys. A: Math. Theor. **42**, 023001 (2009).
  - [11] B. Mirza and H. Mohammadzadeh, Phys. Rev. E **78**, 021127 (2008); Phys. Rev. E **80**, 011132 (2009); Phys. Rev. E **82**, 031137 (2010).
  - [12] A. Stern, Nature (London), **464**, 187 (2010).
  - [13] F.D.M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).
  - [14] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983); B. I. Halperin, *ibid.* **52**, 1583 (1984); R. B. Laughlin, *ibid.* **60**, 2677 (1988); F. Wilczek, *ibid.* **49**, 957 (1982).
  - [15] H. Fehm and M. Stahlsmeier, Phys. Lett. A **250**, 293 (1998).
  - [16] W.-H. Huang, Phys. Lett. A **269**, 333 (2000).
  - [17] S. Guruswamy and K. Schoutens, Nucl. Phys. B **556**, 530 (1999).
  - [18] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991); C. Nayak and F. Wilczek, *ibid.* **479**, 529 (1996); E. Fradkin, C. Nayak and K. Schoutens, *ibid.* **546**, 711 (1999); P. Bouwknegt and K. Schoutens, *ibid.* **547**, 501 (1999); X. G. Wen, Phys. Rev. Lett. **66**, 802 (1991); K. Schoutens, *ibid.* **79**, 2608 (1997); K.

- Schoutens, *ibid.* **81**, 1929 (1998); N. Read and E. Rezayi, Phys. Rev. B **54**, 16864 (1996); N. Read and E. Rezayi, *ibid.* **59**, 8084 (1999); X. Wan, Z.-X. Hu, E. H. Rezayi and K. Yang, *ibid.* **77**, 165316 (2008).
- [19] A. Kitaev, Ann. Phys. (N.Y.) **303**, 2 (2003); R. Raussendorf and J. Harrington, Phys. Rev. Lett. **98**, 190504 (2007); S. Das Sarma, M. Freedman and C. Nayak, *ibid.* **94**, 166802 (2005).